"**Dissipation, Fluctuations, Chaos and Quntum Systems**" was written in December of 2001. It was meant to be a review of my group's work on Quantum Chaos and was dedicated to my late and dear friend, Joel E. Keizer. Joel and I were scientists in that second echelon of scientists who are not famous celebrities, even though of competence and with success in our work. We did our science in the University system where it is practiced by tens of thousands of our peers. We were able to create collaborations with students, post-docs and faculty members that were a benefit scientifically and satisfying personally. In doing this work, before 2001 and since then, I was very fortunate in having collaborations with truly outstanding colleagues. My student years had exposed me to the best scientists out there and these individuals compared very favorably. There were three Ph.D. students: Timothy C. Elston, William Mather and Mee-Hyang Choi; one Joseph Ford post-doctoral Fellow: Luz Vela-Arealo; and two faculty colleagues: Joel E. Keizer and Rajarshi Roy. Without them much of this work would not have been done.

When I rediscovered this forgotten manuscript in September of 2012, I realized it should be on the web. I think the results deserve more recognition than they have as yet received. Two papers in particular merit broader appreciation: Coherent states of the driven Rydberg atom: Quantum-classical correspondence of periodically driven systems, Luz V. Vela-Arevalo and R. F. Fox, Physical Review A 71, 063403 (2005), 12 pages. This paper has been selected by the Editor of and appears in Virtual Journal of Ultrafast Science-July 2005, Volume 4, Issue 7. Coherent State Analysis of the Quantum Bouncing Ball, William Mather and R.F. Fox, Physical Review A, 73 032109 (2006), 9 pages.

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Dissipation, Fluctuations, Chaos and Quantum Systems

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Abstract

In 1990, two papers appeared [1,2] in which it was shown that 1) there is a fundamental connection between the dynamics of fluctuations and chaos in macroscopic classical dissipative systems; and 2) this connection is paralleled in quantum systems in the sense that the classical Lyapunov exponent is a quantum signature of classical chaos. Several examples of the amplification of intrinsic fluctuations by chaos in classical dissipative systems are given in support of 1). In support of 2), the quantum signature of classical chaos is established using Husimi-Wigner distributions that have precise classical analogues in classical phase space. (Related effects exist for discrete maps when described by the Frobenius-Perron operator perspective.) The best way to construct these Husimi-Wigner distributions is to use generalized coherent states. Following a development by John Klauder, generalized Gaussian coherent states were constructed. These new coherent states have applications for systems as diverse as Rydberg atom electrons and a particle in a box. This presentation reviews the progression of ideas that led to these results.

Introduction

A fundamental connection exists between the dynamics of fluctuations in macroscopic dissipative systems and the characterization of such systems as chaotic. This connection is created by the time evolving Jacobi matrix [1,2]. For such systems, chaos is defined by the existence of a positive largest Lyapunov exponent. The existence of such a positive exponent is determined by the time evolving Jacobi matrix. As the system trajectory evolves in time, the instantaneous Jacobi matrix determines the local linearized manifolds of stability and instability. As the trajectory evolves so does the Jacobi matrix and the tendency towards instability is averaged over time, yielding a positive largest Lyapunov exponent for a chaotic system [2]. Macroscopic systems also possess intrinsic fluctuations because of their molecular constituency and because they have a non-zero temperature. It is possible to generate a general framework for such systems in which one starts with a master equation. In an appropriate macroscopic limit, the master equation can be reduced to a Fokker-Planck equation [2], or to an equivalent Langevin description. Within this framework, one obtains a deterministic system of equations for the macroscopic values of the variables, and an auxiliary equation for the time evolution of the covariance matrix for the macrovariable fluctuations. This equation is

$$\frac{d}{dt}\mathbf{C} = \mathbf{JC} + \mathbf{CJ}^{+} + \mathbf{R}$$

in which $\mathbf{C}$ is the fluctuation covariance matrix, $\mathbf{J}$ is the Jacobi matrix ($\mathbf{J}^{+}$ is its adjoint) and $\mathbf{R}$ is a symmetric matrix related to the strength of the fluctuating force correlations of the Langevin description. This equation is valid for the linearized fluctuation regime. It follows that if the deterministic dynamics of this macrovariable system is chaotic then the initial growth rate of the fluctuation covariances is exponential with an exponent that is twice the largest Lyapunov exponent [2]. Eventually the growth will be sufficiently large that it will become quenched by nonlinearities in the macroscopic description. However, the initial exponential stage of growth implies that a measurement of the covariances will determine the largest Lyapunov exponent.

All of this can be paralleled for the quantum case. The analogue to the deterministic equations is the classical Hamilton equations of motion and the analogue to the intrinsic fluctuations is the quantum noise created by the existence of a non-zero Planck's constant. These loose notions are formalized by the connection between quantum mechanics and classical phase space provided by the Wigner distribution [2], the time evolution of which is the analogue to the master equation for macroscopic classical systems. In this case the time evolution of the quantum covariances, $\mathbf{C}$, is

$$\frac{d}{dt}\mathbf{C} = \mathbf{JC} + \mathbf{CJ}^{+}$$

wherein the Jacobi matrix is that determined by the classical Hamiltonian equations. There is no analogue to $\mathbf{R}$ for the non-dissipative quantum case to create a non-zero $\mathbf{C}$ from an initially vanishing $\mathbf{C}$ as in the macrovariable case. However, the covariance matrix does not vanish initially here because of the Heisenberg uncertainty principle. If the corresponding classical dynamics is chaotic then the initial growth rate of the quantum covariances will be exponential and the exponent will equal twice the largest classical Lyapunov exponent [2]. Thus a measurement of the initial growth rate of the quantum covariances determines the largest classical Lyapunov exponent. In this sense, the Lyapunov exponent of the classical motion is a quantum signature of chaos. In a more refined approach, it is the Husimi-Wigner distribution that is used to make the connection with classical phase space. These distributions bring coherent states into the picture.

Macrovariable examples

In our first demonstration of the connection between chaos and amplification of intrinsic fluctuations [1] we numerically studied the Lorenz model viewed as a Galerkin truncation of the hydrodynamic equations for the Rayleigh-Benard problem. Since it is known how to describe the intrinsic hydrodynamic fluctuations for the Navier-Stokes equations [3], unambiguous fluctuation terms can be deduced for the Lorenz model with no new free parameters. We showed an amplification of the hydrodynamic fluctuation correlation by about a thousand-fold before nonlinear quenching took over. This amplification smeared out fine details in the attractor. A more refined account was given later [4]. In subsequent work [5], we showed that this effect on the attractor was general. In this paper, the Rossler model was studied and smoothing of a fractal attractor was demonstrated. We also treated the Josephson junction in which the intrinsic noise is Johnson noise. Here again the effect of smoothing on the attractor was nearly visible at the macroscopic scale.

An attempt was made to see these effects experimentally in a chaotic multimode laser system [6]. Many orders of magnitude of amplification was observed. However, certain technical difficulties prevented an absolutely definitive claim of success. In an attempt to find a better experimental system, a numerical study of the ammonia laser was performed [7]. In this case phase fluctuation amplification by two orders of magnitude was predicted from a numerical analysis of nonlinear Langevin equations. The experiment has not yet been done. Nevertheless, as was originally shown by Haken et al. [8], the description of the ammonia laser can be reduced to the Lorenz equations. The fluctuations in this case are totally different from those for the truncation of the Rayleigh-Benard problem and include multiplicative noise terms. This enabled us to see the difference between two system that had identical deterministic descriptions but very different intrinsic fluctuations.

Another perspective on amplification of fluctuations by chaos is afforded by the study of chemical reaction systems. It is possible to study these systems from a master equation point of view. When the covariances of the fluctuations are amplified, the distributions described by the master equation broaden. This broadening can be sufficiently great that a contraction down to the usual mass action reaction equations is no longer valid. Hence chaos can cause a breakdown of the validity of the mass action equations when those same deterministic equations predict chaos. This has been extensively studied by Ray Kapral et al. [9]. In their paper, they refer to our work and to their observation of strange attractor dephasing and smoothing, and say: " Fluctuations do give rise to significant effects on the scale of the attractor size in phase space."

Husimi-Wigner distributions

In 1932, Wigner [10] introduced his famous distribution function and an equation for its time evolution. He returned to this topic in 1981 [11]. In these papers, he demonstrated a direct connection with the Liouville equation for classical dynamics in classical phase space. Since classical chaos is couched in the language of phase space trajectories, it is often thought that no direct correspondence with quantum mechanics is possible. Quantum signatures of classical chaos have to do with energy eigenstate distributions [12] (also associated with the Wigner name) or with Heller scars in eigenstate profiles [13]. Obtaining a signature related to the classical Lyapunov exponent does not seem to be in the cards. The Wigner distribution changes this erroneous perspective. However, there is still a technical catch. The Wigner distribution is not a true distribution because it is not always non-negative everywhere in phase space. Thus, it does not lead to an unambiguous corresponding distribution for classical phase space. This problem was remedied in 1940 by Husimi [14].

Husimi-Wigner distributions are Wigner distributions that have been smoothed in an appropriate manner so that they are everywhere non-negative. They yield directly an interpretation as ensemble distributions in classical phase space. In a paper on the kicked pendulum [15] we showed how Gaussian smoothing according to Husimi and O'Connell-Wigner yielded a precise correspondence between the time evolution of quantum variances and classical ensemble variances. The initial Husimi-Wigner distribution corresponds with an initial Gaussian ensemble in classical phase space. Simultaneously the variances of both can be evolved in time. For chaotic parameter values, as far as the classical rendering of the model is concerned, the classical ensemble variances and the quantum variances initially grow exponentially at precisely the same rates. Moreover, even after nonlinearities quench the exponential growth, the correspondence of temporal behavior for both kinds of variances remains the same for some time. There is another technical point that needs to be emphasized here. Both the Husimi-Wigner distribution and the corresponding classical phase space ensemble are initially localized somewhere in phase space. Thus, one does not get the global largest Lyapunov exponent from the growth rate of the variances but instead gets the local Lyapunov exponent. Nonlinear quenching occurs before exploration of the entire phase space can occur, as would be required in order to get the global Lyapunov exponent.

Coherent States

The Gaussian smoothing of Husimi can be achieved in a much more elegant manner, as was observed by Chang and Shi in 1986 [16]. Instead of a brute force Gaussian average, one simply takes the wavefunction for the Wigner distribution and makes an inner product of it with a harmonic oscillator coherent state. If we label the coherent state by the complex number α, then a classical p and q for the oscillator can be made from the real and imaginary parts of α. The classical phase space ensemble distribution and the quantum Husimi-Wigner distribution are both given by

$$D_{HW}(p,q) = \frac{1}{\pi} |<\alpha|\psi>|^2$$

where ψ is the wavefunction for the construction of the Wigner distribution. This structure is manifestly non-negative and happens to be normalized to unity.

Spurred on by this construction, we next looked at the kicked top. The pendulum is planar, moving in two dimensions, but the top moves in three dimensions. It was clear that one should use generalized, su(2) coherent states to make the Husimi-Wigner distributions in this case. This works beautifully and results comparable to those obtained for the pendulum were produced [17]. It was shown that as the total angular momentum was increased relative to Planck's constant, the length of time for pure exponential growth of the variance also increased. It was also seen that the correspondence of the time evolution of the variances from the classical and the quantum points of view persisted well beyond the nonlinear saturation of the exponential phase of the growth. However, the distributions themselves began to show significant differences. For the classical case, the ensemble is made up of non-interacting points whereas for the quantum case, the wave-packet can interfere with itself. The coordinate in this case is an angle between 0 and 2π. Thus, by the time exponential growth of the distributions increased the angular variance to the point of filling the finite interval available, the quantum case showed interference patterns in the wave function whereas the classical ensemble approached an invariant attractor. All of this is to be expected.

One lesson learned about quantum-classical correspondence from these studies was that it is not desirable to try to make quantum mechanics correspond with sharp, point-wise classical trajectories. Rather, the natural correspondence is with classical phase space ensembles that are not as sharp as Dirac delta functions. Indeed, Dirac delta functions are not stable objects for chaotic dynamics. If a Dirac delta function on phase space is considered to be the limit of a Gaussian distribution, and is replaced by a Gaussian with a non-zero standard deviation, then chaos will exponentially amplify the standard deviation, i. e, the variances as we have been discussing. Thus the Gaussian distribution will blow up. We made an excursion into the study of this phenomenon by looking at discrete maps using the Frobenius-Perron equation [18,19]. Here, the Bernoulli and Bakers maps were explored and eigenfunctions in the sense of distributions were constructed. Point-wise trajectories for these classical systems were found to be unstable.

The question raised by this development is whether one can go further and study more complicated systems. To do so requires a richer variety of generalized coherent states for the construction of Husimi-Wigner distributions. Beyond the harmonic oscillator coherent states and the su(2) generalized coherent states there is a large literature of conflicting approaches. I found the approach of John Klauder most attractive [20]. I will close this review with an account of what was discovered.

Generalized Gaussian Klauder Coherent States

The problem of generalizing the concept of coherent states from the harmonic oscillator to other systems was already broached by Schrodinger in his 1926 paper in which he introduced harmonic oscillator coherent states [21]. The case of the Coulomb potential was raised by him. He was unsuccessful with this goal. In modern times, this problem has been extended to Rydberg atoms generally, but only recently. The first real attempt at coherent states for the Rydberg problem was made in 1997 [22]. I was blissfully unaware of this paper when I began my own approach to the problem by following the lead of Klauder [20]. Since I knew about the Lenz-Runge vector and how it could be used to convert the algebra of the Rydberg problem into a problem with algebra su(2) x su(2), and I had already worked out generalized coherent states for su(2), I knew I could solve this problem. My paper appeared in 1999 [23]. One amusing byproduct of the approach was a purely quantum mechanical derivation of Kepler's third law that is intimately tied to the Bohr correspondence principle.

The generalized Klauder coherent states constructed for the Rydberg problem have the property that they describe a wave packet that is highly localized in an orbital sense. Initially, the wave packet describes a lump of probability located somewhere on what would appear to be a classical orbit, much like a planet in its orbit. The key point here is that it is not a point but a lump (see above). This lump moves around the orbit as if it were a classical object. However, there was a catch, both for my initial construction [23] and for the earlier one [22]. The localization in the azimuthal angle deteriorated rapidly. The other orbital features were robustly retained, but the azimuthal phase lost its coherence quickly, in less than one period. I knew this as I wrote my paper and in the last section I introduced a resolution to this difficulty, the generalize Gaussian Klauder coherent states. With these state it is possible to create a lump that retains its azimuthal coherence for many orbits, albeit slowly losing this coherence with time. The larger the principal quantum number is, the longer azimuthal coherence is retained.

The next question was: how general were these states? We first had to look at how they compared with the standard coherent states for the harmonic oscillator when applied to a harmonic oscillator. The generalized Gaussian Klauder coherent states have an extra parameter compared with the standard coherent states and when this parameter is large, the generalized states converge on the standard states. Thus, they provide a richer set of states, even for the oscillator. For a planar rotor, they produce a quantum wave packet that describes a lump of probability behaving like a classical rotor for many vibrations or rotations. Eventually these states decohere but this can be delayed as long as one likes by adjusting parameters. Even a particle in a one dimensional box can be described with these new coherent states. The lump of probability corresponding to the particle bounces back and forth inside the box many times before decoherence is observed, and this too can be delayed arbitrarily long. The centroid of the distribution behave precisely like a classical point particle bouncing between the walls.

One detail remained. Each of these examples involves non-degenerate energy spectra, with the exception of the Rydberg atoms, which are exceptional in another way. The Klauder construction works straight-forwardly for non-degenerate spectra. Crawford [24] showed that if there was an underlying symmetry causing energy spectrum degeneracy then a method developed by Perelomov [25] would get one around this snag. However, a problem as simple as a particle in a two dimensional box has a degenerate energy spectrum caused by number theory, not symmetry. In this case, the degeneracy come from the number of different ways a number can be written as the sum of two squares, an ancient problem in number theory. We found a way to cover this sort of situation that also works for the cases involving symmetry [26].

It remains to apply these new kinds of coherent states to the construction of Husimi-Wigner states. Perhaps this provides a novel approach to quantum-classical correspondence for the Bunimovich stadium problem, and these states may provide a type of wavelet. In addition, it is desirable to include dissipation in the quantum systems. This may require a transition to the density matrix picture for the Husimi-Wigner distributions. These and other problems remain for future research.

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(This review was written in December of 2001. Since then additonal papers were published by my group and can be found in my CV, CVFOX.pdf. September, 2012)

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