Have another piece of Pi

After receiving my simple construction of Pi (π) using Pythagoras' theorem [PiDay], my son, Dan, sent me the paper that is found at [Pi]. In this paper it is shown that in base sixteen there is the Plouffe formula for π :

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n}\left[\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6}\right]$$

At first I thought this formula generated the *hexadecimal digits* for π.
However, the first seven values are given by

$$n = 0: \quad \frac{1}{1}\left[4 - \frac{1}{2} - \frac{1}{5} - \frac{1}{6}\right] = 3.1\overline{3}$$

$$n = 1: \quad \left[\frac{4}{9} - \frac{1}{6} - \frac{1}{13} - \frac{1}{14}\right] = 0.\overline{129426}$$

$$n = 2: \quad \left[\frac{4}{17} - \frac{1}{10} - \frac{1}{21} - \frac{1}{22}\right]$$
$$= 0.0\overline{4222052457346574993633817163228927934810287751 46}$$

$$n = 3: \quad \left[\frac{4}{25} - \frac{1}{14} - \frac{1}{29} - \frac{1}{30}\right] = 0.02\overline{0755336617405 58\ldots}$$

$$n = 4: \quad \left[\frac{4}{33} - \frac{1}{18} - \frac{1}{37} - \frac{1}{38}\right] = 0.0\overline{123137491558544190}$$

$$n = 5: \quad \left[\frac{4}{41} - \frac{1}{22} - \frac{1}{45} - \frac{1}{46}\right] = 0.00\overline{814507749820 58\ldots}$$

$$n = 6: \quad \left[\frac{4}{49} - \frac{1}{26} - \frac{1}{53} - \frac{1}{54}\right] = 0.00\overline{578467155\ldots}$$

in which the over-bar denotes a repeating segment, some of which are very long. These first seven terms already yield an approximation to π that is 3.141587 ... in the standard decimal form. Note that the formula gives directly the coefficient of the n^{th} power of $\frac{1}{16}$ but it is not a *hexadecimal digit*.

The hexadecimal digit expansion for π begins with

3.243F6A8…

This can be interpreted as

$$\pi = \frac{3}{16^0} + \frac{2}{16} + \frac{4}{16^2} + \frac{3}{16^3} + \frac{F}{16^4} + \frac{6}{16^5} + \frac{A}{16^6} + \frac{8}{16^7} + \cdots \equiv \sum_{n=0}^{\infty} \frac{H_n}{16^n}$$

in which we use the standard hexadecimal notation $A = 10$ and $F = 15$. A check of this approximation yields the decimal value

$$3.14159265\ 16056060791015625$$

the first eight decimal digits of which are correct for π.

How do we reconcile these two different expressions? The hexadecimal expansion is

$$\sum_{n=0}^{\infty} \frac{H_n}{16^n}$$

and the Plouffe formula can be written

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n} \left[\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right] \equiv \sum_{n=0}^{\infty} \frac{P_n}{16^n}$$

Obviously the expansion of π in terms of powers of $\frac{1}{16}$ can have many solutions for the coefficients, a_n:

$$\pi = \sum_{n=0}^{\infty} \frac{a_n}{16^n}$$

We can generate the hexadecimal digits directly from π or from the Plouffe formula!

Consider a real number x and the two functions, *integer part*, $I[x]$, and *residual part*, $R[x]$. In the usual decimal expansion, $I[x]$ is the largest integer part of x:

$$I[x] \leq x$$

What is left over is less than 1.0 and is called the residual part of x, $R[x]$:

$$R[x] = x - I[x]$$

Generation of the hexadecimal digits for π can be done iteratively, starting with π:

$$I_0 = I[\pi] = 3$$
$$R_0 = \pi - I_0 = \pi - 3 = 0.14159265 \ldots$$

$$for \quad n \geq 1$$

$$I_n = I[16 \times R_{n-1}]$$
$$R_n = 16 \times R_{n-1} - I_n$$

The n^{th} hexadecimal digit, H_n, is given by

$$H_n = I_n$$

The authors of [Pi] also give an algorithm that uses the Plouffe formula to directly compute the $billionth$ hexadecimal digit of π without having to first compute the preceding $billionth - 1$ hexadecimal digits. I have not verified this algorithm and recognize that the iterative process given above clearly depends on computing all the previous values. Thus, a check of the first few hexadecimal diigits after the billionth hexadecimal digit given in [Pi] cannot be readily checked using my iterative process. On the other hand it would have been nice of the authors of [Pi] to show that their procedure does generate the first eight, say, hexadecinmal digits of π without using iteration.

One may ask why these relations are of interest. Mathematics is a part of human existence closely related to euphoria and joy. Discovering beautiful mathematical formulae is an ecstatic experience, *even* if they are already known to others. As a pastime, mathematics permits endless digressions into minutiae and detail. Not all practitioners are academics and professors. Anyone can develop mathematical skills given enough practice. Evolution has rewarded those organisms that use thinking as a strategy in life by building neural networks that reinforce the successful practice of thinking. These networks reinforce through ecstatic experience, and the neural protein receptors, and their molecular activators, have been identified. Albert Einstein refered to this ecstatic experience as *cosmic religious sense*. I have diecussed this elsewhere in bullet 57 of my Einstein Centennial lecture.

Of interest here is the Plouffe formula. This is named for Simon Plouffe, its human discoverer. The Wiki site contains the curious line:

Plouffe discovered an algorithm for the computation of π in any base in 1996. He has expressed regret for having shared credit for his discovery of this formula with Bailey and Borwein.

This relates to issues of science and integrity.

I have played with a few of the formulae in [Pi]. Let us look at the significance of Plouffe's formula *vis-à-vis* π.

The first interesting formula is:

$$\pi = \int_0^{1/\sqrt{2}} dx \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1 - x^8}$$

Simply using :

$$\frac{1}{1 - x^8} = \sum_{n=0}^{\infty} x^{n8}$$

that is valid for $0 \leq x < 1$ and covers the interval $\left[0, \frac{1}{\sqrt{2}}\right]$ explains this result. Together with a result about integrals given in the paper, one easily arrives at the Plouffe expression:

$$\sum_{n=0}^{\infty} \frac{1}{16^n} \left[\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6}\right]$$

By the substitution $y = \sqrt{2}x$ it can be shown that the integral above gives:

$$\pi = \int_0^1 dy \frac{16y - 16}{y^4 - 2y^3 + 4y - 4}$$

To see this one needs to verify the quite amazing identities:

$$(y^4 - 2y^3 + 4y - 4) \times (y^4 + 2y^3 + 4y^2 + 4y + 4) = y^8 - 16$$

$$(16y - 16) \times (y^4 + 2y^3 + 4y^2 + 4y + 4) = 16y^5 + 16y^4 + 32y^3 - 64$$

Together these two equations imply:

$$\pi = \int_0^1 dy \frac{4 - 2y^3 - y^4 - y^5}{1 - \frac{y^8}{16}}$$

which is where we started when $y = \sqrt{2}x$ is invoked. The truth of:

$$\pi = \int_0^1 dy \frac{16y - 16}{y^4 - 2y^3 + 4y - 4}$$

is established by finding roots and using partial fractions:

$$\frac{16y - 16}{y^4 - 2y^3 + 4y - 4} = \frac{4y}{y^2 - 2} + \frac{8 - 4y}{y^2 - 2y + 2}$$

$$= \frac{2}{y - \sqrt{2}} + \frac{2}{y + \sqrt{2}} + \frac{-2 - 2i}{y - 1 - i} + \frac{-2 + 2i}{y - 1 + i}$$

It is straight forward, albeit subtle, to show:

$$\pi = \int_0^1 dy \left(\frac{2}{y - \sqrt{2}} + \frac{2}{y + \sqrt{2}} + \frac{-2 - 2i}{y - 1 - i} + \frac{-2 + 2i}{y - 1 + i} \right)$$

These are a few of the formulae that arise from Plouffe's formula for π. Many other identities begin from the other results in [Pi].

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