Approximating π without using trigonometric functions

Since March 14 is $\boldsymbol{\pi}$ day, and also my son's birthday, I thought a bit about $\boldsymbol{\pi}$. We know a lot about $\boldsymbol{\pi}$, that it is a transcendental number, and that is plays a central role in trigonometry. I had a hunch that I could approximate $\boldsymbol{\pi}$ arbitrarily well without trigonometric functions by only using Pythagoras' theorem.

$\boldsymbol{\pi}$ is the ratio of the circumference of a circle to its diameter. Imagine a circle and draw two diameters so that they intersect at right angles. Let us suppose that the radius (half the diameter) is of length unity, 1. The circumference would then be $2\boldsymbol{\pi}$. A very crude first approximation to $\boldsymbol{\pi}$ is to connect the four points where the diameters intersect the circumference with four *chords* (straight lines joining two points on the circumference). Each chord, according to Pythagoras' theorem has length $\sqrt{2}$. Draw a picture if you can't see this in your mind. The length of the 4 chords is $4\sqrt{2}$, and since the diameter is 2, the approximation to $\boldsymbol{\pi}$ becomes $2\sqrt{2}$ ~2.828… This is not too close to 3.14159… We have inscribed a square inside the circle by this construction. We can do better as follows. Add two more diameters so that they bisect the right angles between the two existing diameters. The four diameters determine 8 points of intersection on the circumference and connecting adjacent intersection points with chords produces an inscribed regular octagon. Note that the two new diameters also bisect the previously drawn chords that constitute the inscribed square. Several right triangles are formed. Using only Pythagoras' theorem it is straight-forward to show that the length of an octagon chord is $\sqrt{2-\sqrt{2}}$. Since there are now 8 chords and the diameter remains equal to 2 the octagonal approximation to $\boldsymbol{\pi}$ is $4\sqrt{2-\sqrt{2}} \sim 3.0614\ldots$ This is better than for the square but still quite crude. Visually, it is clear that the octagon does a better job approximating the circumference than does the square and it is clear that if we continue to add diameters that bisect the existing angles, and therefore also the existing chords, we will get a better approximation. At each stage of this process we only use Pythagoras' theorem to figure out the lengths of the newly formed chords for a 2^n-gon inscribed inside the circle ($n = 2$ is the square and $n = 3$ is the octagon).

Since it is now clear how the approximation procedure works, we can distill the results by considering the transition from one stage of construction to the next. Suppose we have reached the nth stage in which the chord length is x_n, then the chord length for the next stage, x_{n+1}, can be shown to be given by

$$x_{n+1} = \sqrt{2 - 2\sqrt{1 - \frac{x_n^2}{4}}}$$

This is the straight-forward result from Pythagoras' theorem and the reader is urged to derive it. In this notation, $x_0 = \sqrt{2}$. The approximation to $\boldsymbol{\pi}$ at the nth stage is given by

$$\boldsymbol{\pi_n} = 2^{n+1} x_n$$

It is easy to program the iterative mapping (recurrence formula) for x_n and then multiply by the appropriate power of 2 to get a good approximation to $\boldsymbol{\pi}$. For $n = 5$, the result is

$$\boldsymbol{\pi_5} \sim 3.14127 \ldots$$

which is good to three decimal places. Convergence is slow and we find

$$\boldsymbol{\pi_{10}} \sim 3.14153 \ldots$$

which is good to one more decimal place.

It is also possible to explicitly implement the recurrence formula starting from $x_0 = \sqrt{2}$. The result, that is again easy to verify, is

$$\boldsymbol{\pi} = \lim_{n \to \infty} 2^{n+1} \times \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}$$

in which there are exactly n radicals after the minus sign. This lovely formula is a variation on Viete's formula from the 16th century. It was subsequently derived in the 20th century from a recurrence formula similar to the one above, but based on a trigonometric function identity found by Leonhard Euler in the 18th century. Here we have not used any trigonometric functions whatsoever and have justified all results using only Pythagoras' theorem. That the formula depends only on square roots and 2's is impressive.

A very similar argument is given on page 124 of *What is Mathematics?* by Richard Courant and Herbert Robbins. Their recursion formula and construction are different in detail but independent of trigonometric functions as is the case here.

"There is nothing new under the sun but there are lots of old things we don't know."

Ambrose Bierce, *The Devil's Dictionary*
US author & satirist (1842 - 1914)

Note that for the limit formula to be true the argument of the outer radical must vanish in the limit because the prefactor goes to ∞ in the limit. This means

$$\lim_{n\to\infty}\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}=2$$

in which there are n radicals. This can be proved independently by algebra: let

$$z=\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}$$

Clearly, in the limit, $z^2=2+z$, or equivalently $z^2-z-2=0$. This quadratic equation has two roots, 2 and -1. Only the positive root is a possible solution.

If one starts not with a square but with a hexagon instead and then commenses to iterative bisection then one can verify the formula:

$$\boldsymbol{\pi} = \lim_{n\to\infty} 3 \times 2^n \times \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{3}}}}}$$

in which there are exactly n radicals after the minus sign.

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